

# ON ALGEBRAIC AND MORE GENERAL CATEGORIES WHOSE SPLIT EPIMORPHISMS HAVE UNDERLYING PRODUCT PROJECTIONS

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## Abstract

We characterize those varieties of universal algebras where every split epimorphism considered as a map of sets is a product projection. In addition we obtain new characterizations of protomodular, unital and subtractive varieties as well as varieties of right  $\Omega$ -loops and biternary systems.

## Introduction

It is well known that in the category of groups if

$$0 \longrightarrow K \xrightarrow{\kappa} A \xrightarrow{\alpha} B \longrightarrow 0$$

is a short exact sequence, then  $A$  and  $K \times B$  are bijective as sets, moreover when  $\alpha$  is split, i.e. for each split extension

$$K \xrightarrow{\kappa} A \xrightleftharpoons[\beta]{\alpha} B, \quad \alpha\beta = 1_B, \quad \kappa = \ker(\alpha),$$

this bijection becomes a natural bijection  $K \times B \rightarrow A$  such that the diagram

$$\begin{array}{ccccc} K & \xrightarrow{\langle 1,0 \rangle} & K \times B & \xrightleftharpoons[\langle 0,1 \rangle]{\pi_2} & B \\ \parallel & & \downarrow \varphi & & \parallel \\ K & \xrightarrow{\kappa} & A & \xrightleftharpoons[\beta]{\alpha} & B \end{array}$$

is a morphism of split extensions in the category **Set**, of sets, that is,  $\alpha\varphi = \pi_2$ ,  $\varphi\langle 0,1 \rangle = \beta$ , and  $\varphi\langle 1,0 \rangle = \kappa$ . As shown by E. B. Inyangala, these bijections exists in a more general setting of a variety of right  $\Omega$ -loops (see

[4, 5]), that is, a pointed variety of universal algebras  $\mathcal{V}$  with constant 0 and binary terms  $x + y$  and  $x - y$  satisfying the identities:

$$x + 0 = x \quad (1)$$

$$x - x = 0 \quad (2)$$

$$(x + y) - y = x \quad (3)$$

$$(x - y) + y = x \quad (4)$$

Moreover, he showed that if a pointed variety  $\mathcal{V}$  with constant 0 has binary terms  $x + y$  and  $x - y$  and there exist bijections (as above) constructed (in the same way as for groups) using those terms, i.e.  $\varphi(k, b) = \kappa(k) + \beta(b)$  and  $\varphi^{-1}(a) = (\lambda(a), \alpha(a))$ , where  $\lambda$  is the unique map such that  $\kappa\lambda(a) = a - \beta\alpha(a)$ , then  $\mathcal{V}$  is a variety of right  $\Omega$ -loops and in particular the identities (1) - (4) hold for  $x + y$  and  $x - y$ . In this paper we prove that if for a pointed variety  $\mathcal{V}$  there exist natural bijections as above, then  $\mathcal{V}$  is a variety of right  $\Omega$ -loops (see Theorem 2.1).

For any category  $\mathbb{C}$  let  $\mathbf{Pt}(\mathbb{C})$  to be the category of split epimorphisms in  $\mathbb{C}$ : an object is a quadruple  $(A, B, \alpha, \beta)$  where  $A$  and  $B$  are objects in  $\mathbb{C}$  and  $\alpha : A \rightarrow B$  and  $\beta : B \rightarrow A$  are morphisms in  $\mathbb{C}$  with  $\alpha\beta = 1_B$ ; a morphism  $(A, B, \alpha, \beta) \rightarrow (A', B', \alpha', \beta')$  is a pair of morphisms  $(f : A \rightarrow A', g : B \rightarrow B')$  such that in the diagram

$$\begin{array}{ccc} A & \xrightleftharpoons[\beta]{\alpha} & B \\ f \downarrow & & \downarrow g \\ A' & \xrightleftharpoons[\beta']{\alpha'} & B' \end{array}$$

$\alpha'f = g\alpha$  and  $f\beta = \beta'g$ . Throughout this paper for any objects  $A$  and  $B$  we will denote by  $\pi_1$  and  $\pi_2$  the first and second product projections respectively. We will use the same notation for the first and second pullback projections and will write

$$(A \times_{\langle f, g \rangle} B, \pi_1, \pi_2)$$

for the pullback of  $f : A \rightarrow C$  and  $g : B \rightarrow C$  as in the diagram

$$\begin{array}{ccc} A \times_{\langle f, g \rangle} B & \xrightarrow{\pi_2} & B \\ \pi_1 \downarrow & & \downarrow g \\ A & \xrightarrow{f} & C. \end{array}$$

For any morphisms  $u : W \rightarrow A$  and  $v : W \rightarrow B$  with  $fu = gv$  we will write

$$\langle u, v \rangle : W \rightarrow A \times_{\langle f, g \rangle} B$$

for the unique morphism with  $\pi_1\langle u, v \rangle = u$  and  $\pi_2\langle u, v \rangle = v$ .

We prove that for a pointed variety  $\mathcal{V}$ , if for each  $(A, B, \alpha, \beta)$  in  $\mathbf{Pt}(\mathcal{V})$  there exists a natural bijection  $\varphi : K \times B \rightarrow A$ , where  $\kappa : K \rightarrow A$  is the kernel of  $\alpha$ , such that the diagram

$$\begin{array}{ccc} K \times B & \xrightleftharpoons[\langle 0, 1 \rangle]{\pi_2} & B \\ \varphi \downarrow & & \parallel \\ A & \xrightleftharpoons[\beta]{\alpha} & B \end{array}$$

is a morphism in  $\mathbf{Pt}(\mathbf{Set})$ , then  $\mathcal{V}$  is a variety of right  $\Omega$ -loops (see Corollary 2.2 ). There is a *natural* generalization of this condition for any variety  $\mathcal{V}$ , namely asking for each  $(A, B, \alpha, \beta)$  in  $\mathbf{Pt}(\mathcal{V})$  and for each morphism  $f : E \rightarrow B$  that there exists a bijection

$$\varphi : (A \times_{\langle \alpha, f \rangle} E) \times B \rightarrow E \times A$$

natural in both  $(A, B, \alpha, \beta)$  and  $f : E \rightarrow B$ , such that the diagram

$$\begin{array}{ccc} (A \times_{\langle \alpha, f \rangle} E) \times B & \xrightleftharpoons[\langle \beta f, 1 \rangle \times 1]{\pi_2 \times 1} & E \times B \\ \varphi \downarrow & & \parallel \\ E \times A & \xrightleftharpoons[1 \times \beta]{1 \times \alpha} & E \times B \end{array}$$

is a morphism in  $\mathbf{Pt}(\mathbf{Set})$ . It is clear that for a pointed variety this condition implies the previous condition, since taking  $E$  to be the zero object and  $f$  to be the unique morphism from  $E$  to  $B$  makes

$$\pi_1 : A \times_{\langle \alpha, f \rangle} E \rightarrow A$$

the kernel of  $\alpha$ . In Section 4 we prove that this condition is equivalent to the same condition under the restriction that each  $f$  as above is an identity morphism (see Theorem 4.6). We also prove that a variety satisfies this condition if and only if it is a *biternary system* [7] that is there exist ternary terms  $p(x, y, z)$  and  $q(x, y, z)$  satisfying the identities

$$p(x, x, y) = y \tag{5}$$

$$p(q(x, y, z), z, y) = x = q(p(x, y, z), z, y). \tag{6}$$

However, there are other generalizations that may be considered. In a variety  $\mathcal{V}$  with constants, for each  $X$ , let  $\theta_X : 1 \rightarrow X^n$  be a map (natural in  $X$ ) such that the composite with each product projection  $\pi_i : X^n \rightarrow X$

gives a constant. We could then consider the following condition: for each  $(A, B, \alpha, \beta)$  in  $\mathbf{Pt}(\mathcal{V})$  there exists a natural split epimorphism (in the category of sets)

$$\varphi : (A^n_{\langle \alpha^n, \theta_B \rangle} 1) \times B \rightarrow A$$

with splitting

$$\psi : A \rightarrow (A^n_{\langle \alpha^n, \theta_B \rangle} 1) \times B$$

such that in the diagram

$$\begin{array}{ccc} (A^n_{\langle \alpha^n, \theta_B \rangle} 1) \times B & \xrightleftharpoons[\langle \langle \theta_A, 1 \rangle!_B \rangle \times 1]{\pi_2} & B \\ \psi \updownarrow \varphi & & \parallel \\ A & \xrightleftharpoons[\beta]{\alpha} & B \end{array}$$

the upward and downward directed sub-diagrams are morphisms in  $\mathbf{Pt}(\mathbf{Set})$ . We prove in Section 3 that this condition is equivalent to  $\mathcal{V}$  being a proto-modular variety [2] of *type*  $n$ , that is, a variety  $\mathcal{V}$  with constants  $e_1, \dots, e_n$ , binary terms  $s_1(x, y), \dots, s_n(x, y)$  and an  $n + 1$ -ary term  $p(x_1, \dots, x_n, z)$  satisfying the identities:

$$s_i(x, x) = e_i \quad i \in \{1, \dots, n\} \quad (7)$$

$$p(s_1(x, z), \dots, s_n(x, z), z) = x. \quad (8)$$

Note that requiring  $\varphi$  to be a bijection gives the addition conditions

$$s_i(p(x_1, \dots, x_n, y), y) = x_i \quad \text{for all } i \in 1, \dots, n. \quad (9)$$

In order to study these conditions simultaneously we make a further generalization described in Section 1.

## 1 The general setting

In this section we replace a forgetful functor from a variety into the category of sets (or pointed sets) with an abstract functor (satisfying certain conditions) and consider a generalization allowing us to study simultaneously both generalizations discussed in the introduction.

For a set  $\mathbf{n}$ , a category  $\mathbb{D}$  with finite products and products indexed over  $\mathbf{n}$ , and for functors  $F, G, H : \mathbb{C} \rightarrow \mathbb{D}$  we denote by  $F^{\mathbf{n}}$  the  $\mathbf{n}$  indexed product of  $F$  with itself and by  $G \times H$  the product of  $G$  and  $H$  in the functor category  $\mathbb{D}^{\mathbb{C}}$ .

Throughout this section we will assume that:

1.  $\mathbb{A}$  is a category with finite products;
2.  $\mathbf{m}$  and  $\mathbf{n}$  are sets;
3.  $\mathbb{X}$  is a category with finite limits and products indexed by the sets  $\mathbf{m}$  and  $\mathbf{n}$ ;
4.  $U : \mathbb{A} \rightarrow \mathbb{X}$  is a functor preserving finite products;
5.  $\theta : U^{\mathbf{m}} \rightarrow U^{\mathbf{n}}$  is a natural transformation.

Let  $\Delta : \mathbb{A} \rightarrow \mathbf{Pt}(\mathbb{A})$  be the functor sending  $X$  in  $\mathbb{A}$  to  $(X \times X, X, \pi_2, \langle 1, 1 \rangle)$  and let  $D_{\mathbb{A}}$  be the functor  $\mathbf{Pt}(\mathbb{A}) \rightarrow \mathbb{A}$  taking  $(A, B, \alpha, \beta)$  to  $B$ . Let  $V : \mathbf{Pt}(\mathbb{A}) \rightarrow \mathbf{Pt}(\mathbb{X})$  and  $W : \mathbf{Pt}(\mathbb{A}) \rightarrow \mathbf{Pt}(\mathbb{X})$  be the functors sending  $(A, B, \alpha, \beta)$  in  $\mathbf{Pt}(\mathbb{A})$  to

$$((U(A)^{\mathbf{n}})_{\langle U(\alpha)^{\mathbf{n}}, \theta_B \rangle}^{\times} U(B)^{\mathbf{m}} \times U(B), U(B)^{\mathbf{m}} \times U(B), \pi_2 \times 1, \langle U(\beta)^{\mathbf{n}} \theta_B, 1 \rangle \times 1)$$

and

$$(U(B)^{\mathbf{m}} \times U(A), U(B)^{\mathbf{m}} \times U(B), 1 \times U(\alpha), 1 \times U(\beta))$$

respectively.

From the beginning of the next section we will consider the case where  $\mathbb{A}$  is a variety,  $\mathbb{X}$  is the category of sets,  $U$  is the usual forgetful functor from the variety to the category of sets,  $\mathbf{m} = \{1, \dots, m\}$ ,  $\mathbf{n} = \{1, \dots, n\}$ , and  $\theta$  is constructed from  $n$   $m$ -ary terms of  $\mathbb{A}$ . In particular when  $\mathbb{A}$  is pointed with constant  $0$ ,  $\mathbf{n} = \{1\}$ ,  $\mathbf{m} = \emptyset$ , and  $\theta : U^{\mathbf{m}} \rightarrow U^{\mathbf{n}}$  is the natural transformation with component at  $X$   $\theta_X(1) = 0$  (where  $1$  is the unique element in  $U^{\mathbf{m}}(X)$ ), it can be seen that

$$\pi_1 : U(A)^{\mathbf{n}}_{\langle U(\alpha)^{\mathbf{n}}, \theta_B \rangle}^{\times} U(B)^{\mathbf{m}} \rightarrow U(A)$$

is up to isomorphism the image under  $U$  of the kernel of  $\alpha$  and the bijections mentioned at the start of the introduction become components of a natural transformation  $V \rightarrow W$ .

**Lemma 1.1.** *Each of the following types of data uniquely determine each other:*

- (a) a natural transformation  $\tau : V \rightarrow W$ ;
- (b) a natural transformation  $\bar{\tau} : V\Delta \rightarrow W\Delta$ ;
- (c) natural transformations  $\rho : (U^{\mathbf{n}} \times U^{\mathbf{m}}) \times U \rightarrow U$  and  $\zeta : U^{\mathbf{m}} \times U \rightarrow U^{\mathbf{m}}$ ;

*Proof.* For each  $(A, B, \alpha, \beta)$  in  $\mathbf{Pt}(\mathbb{A})$  and  $X$  in  $\mathbb{A}$ , let  $(\varphi_{1(A,B,\alpha,\beta)}, \varphi_{0(A,B,\alpha,\beta)}) = \tau_{(A,B,\alpha,\beta)}$  and  $(\bar{\varphi}_{1X}, \bar{\varphi}_{0X}) = \bar{\tau}_X$ . The diagram

$$\begin{array}{ccc}
 P_X \times U(X) & \xrightarrow{\bar{\varphi}_{1X}} & U(X)^{\mathbf{m}} \times U(X \times X) \\
 \swarrow \langle U(\pi_1)^{\mathbf{n}} \pi_1, \pi_2 \rangle \times 1 & \uparrow \parallel & \swarrow 1 \times \langle U(\pi_1), U(\pi_2) \rangle \\
 (U(X)^{\mathbf{n}} \times U(X)^{\mathbf{m}}) \times U(X) & \xrightarrow{p_X} & U(X)^{\mathbf{m}} \times (U(X) \times U(X)) \\
 \swarrow \langle U(1,1)^{\mathbf{n}} \theta_X, 1 \rangle \times 1 & \uparrow \parallel & \swarrow 1 \times U(1,1) \\
 \searrow \pi_2 \times 1 & \downarrow \parallel & \searrow 1 \times U(\pi_2) \\
 \searrow \langle \theta_X, 1 \rangle \times 1 & \downarrow \parallel & \searrow 1 \times \pi_2 \\
 U(X)^{\mathbf{m}} \times U(X) & \xrightarrow{\bar{\varphi}_{0X}} & U(X)^{\mathbf{m}} \times U(X),
 \end{array}$$

in which

$$P_X = U(X \times X)^{\mathbf{n}} \times_{\langle U(\pi_2)^{\mathbf{n}}, \theta_X \rangle} U(X)^{\mathbf{m}}$$

and

$$p_X = \langle \zeta_X(\pi_2 \times 1), \langle \rho_X, \rho_X(\langle \theta_X \pi_2, \pi_2 \rangle \times 1) \rangle \rangle,$$

is a commutative diagram of morphisms in  $\mathbf{Pt}(\mathbb{X})$ , and shows the relationship between  $\bar{\tau}$  and  $\rho$  and  $\zeta$ . The commutative diagrams

$$\begin{array}{ccc}
 (U(A)^{\mathbf{n}} \times_{\langle U(\alpha)^{\mathbf{n}}, \theta_B \rangle} U(B)^{\mathbf{m}}) \times U(B) & \xrightarrow{\varphi_{1(A,B,\alpha,\beta)}} & U(B)^{\mathbf{m}} \times U(A) \\
 \downarrow \langle \pi_1, U(\beta)^{\mathbf{m}} \pi_2 \rangle \times U(\beta) & \downarrow (U(\langle 1, \beta \alpha \rangle)^{\mathbf{n}} \times U(\beta)^{\mathbf{m}}) \times U(\beta) & \downarrow U(\beta)^{\mathbf{m}} \times U(\langle 1, \beta \alpha \rangle) \\
 (U(A \times A)^{\mathbf{n}} \times_{\langle U(\pi_2)^{\mathbf{n}}, \theta_A \rangle} U(A)^{\mathbf{m}}) \times U(A) & \xrightarrow{\varphi_{1\Delta(A)} = \bar{\varphi}_{1A}} & U(A)^{\mathbf{m}} \times U(A \times A) \\
 \downarrow \langle U(\pi_1)^{\mathbf{n}} \pi_1, \pi_2 \rangle \times 1 & & \downarrow U(\alpha)^{\mathbf{m}} \times U(\pi_1) \\
 (U(A)^{\mathbf{n}} \times U(A)^{\mathbf{n}}) \times U(A) & \xrightarrow{\langle U(\alpha)^{\mathbf{m}} \zeta_A(\pi_2 \times 1), \rho_A \rangle} & U(B)^{\mathbf{m}} \times U(A)
 \end{array}$$
  

$$\begin{array}{ccc}
 U(B)^{\mathbf{m}} \times U(B) & \xrightarrow{\varphi_{0(A,B,\alpha,\beta)}} & U(B^{\mathbf{m}} \times U(B)) \\
 \downarrow U(\beta)^{\mathbf{m}} \times U(\beta) & & \downarrow U(\beta)^{\mathbf{m}} \times U(\beta) \\
 U(A)^{\mathbf{m}} \times U(A) & \xrightarrow{\varphi_{0\Delta(A)}} & U(A)^{\mathbf{m}} \times U(A) \\
 \downarrow U(\alpha)^{\mathbf{m}} \times U(\alpha) & & \downarrow U(\alpha)^{\mathbf{m}} \times U(\alpha) \\
 U(B)^{\mathbf{m}} \times U(B) & \xrightarrow{\varphi_{0\Delta(B)}} & U(B)^{\mathbf{m}} \times U(B) \\
 & \searrow \langle \zeta_B, \rho_B(\langle \theta_B, 1 \rangle \times 1) \rangle & \parallel \\
 & & U(B)^{\mathbf{m}} \times U(B)
 \end{array}$$

show the relationships between  $\tau$  and  $\bar{\tau}$ , and  $\tau$  and  $\rho$  and  $\zeta$ .  $\square$

**Lemma 1.2.** *Each of the following types of data uniquely determine each other:*

- (a) a natural transformation  $\gamma : W \rightarrow V$ ;
- (b) a natural transformation  $\bar{\gamma} : W\Delta \rightarrow V\Delta$ ;
- (c) natural transformations  $\sigma : U^{\mathbf{m}} \times (U \times U) \rightarrow U^{\mathbf{n}}$ ,  $\eta : U^{\mathbf{m}} \times U \rightarrow U^{\mathbf{m}}$  and  $\epsilon : U^{\mathbf{m}} \times U \rightarrow U$  with components at each  $X$  in  $\mathbb{A}$  making the diagram

$$\begin{array}{ccc} U(X)^{\mathbf{m}} \times U(X) & \xrightarrow{1 \times \langle 1, 1 \rangle} & U(X)^{\mathbf{m}} \times (U(X) \times U(X)) \\ \eta_X \downarrow & & \downarrow \sigma_X \\ U(X)^{\mathbf{m}} & \xrightarrow{\theta_X} & U(X)^{\mathbf{n}} \end{array} \quad (10)$$

commute.

*Proof.* For each  $(A, B, \alpha, \beta)$  in  $\mathbf{Pt}(\mathbb{A})$  and  $X$  in  $\mathbb{A}$ , let  $(\psi_{1(A, B, \alpha, \beta)}, \psi_{0(A, B, \alpha, \beta)}) = \gamma_{(A, B, \alpha, \beta)}$  and  $(\bar{\psi}_{1_X}, \bar{\psi}_{0_X}) = \bar{\gamma}_X$ . The diagram

$$\begin{array}{ccccc} & & P_X \times U(X) & \xleftarrow{\bar{\psi}_{1_X}} & U(X)^{\mathbf{m}} \times U(X \times X) \\ & \swarrow \langle U(\pi_1)^{\mathbf{n}} \pi_1, \pi_2 \rangle \times 1 & \uparrow \parallel & \searrow 1 \times \langle U(\pi_1), U(\pi_2) \rangle & \uparrow \parallel \\ (U(X)^{\mathbf{n}} \times U(X)^{\mathbf{m}}) \times U(X) & \xleftarrow{q_X} & U(X)^{\mathbf{m}} \times (U(X) \times U(X)) & & \\ \swarrow \langle U \langle 1, 1 \rangle^{\mathbf{n}} \theta_X, 1 \rangle \times 1 & \uparrow \parallel \pi_2 \times 1 & \searrow 1 \times U \langle 1, 1 \rangle & \uparrow \parallel 1 \times U(\pi_2) & \\ & \downarrow \parallel & & \downarrow \parallel & \\ & U(X)^{\mathbf{m}} \times U(X) & \xleftarrow{\bar{\psi}_{0_X}} & U(X)^{\mathbf{m}} \times U(X), & \\ \swarrow \langle \theta_X, 1 \rangle \times 1 & \uparrow \parallel & \searrow 1 \times \langle 1, 1 \rangle & \uparrow \parallel & \\ & U(X)^{\mathbf{m}} \times U(X) & & U(X)^{\mathbf{m}} \times U(X) & \end{array}$$

in which

$$P_X = U(X \times X)^{\mathbf{n}} \times_{\langle U(\pi_2)^{\mathbf{n}}, \theta_X \rangle} U(X)^{\mathbf{m}}$$

and

$$q_X = \langle \langle \sigma_X, \eta_X(1 \times \pi_2) \rangle, \epsilon_X(1 \times \pi_2) \rangle,$$

is a commutative diagram of morphisms in  $\mathbf{Pt}(\mathbb{X})$ , and shows the relationship between  $\bar{\gamma}$  and  $\sigma$ ,  $\eta$  and  $\epsilon$ . The equations

$$\gamma_{\Delta(X)} = \bar{\gamma}_X$$

and

$$\psi_{1(A, B, \alpha, \beta)} = \langle \langle \sigma_A(U(\beta)^{\mathbf{m}} \times U(\langle 1, \beta \alpha \rangle)), \eta_B(1 \times U(\alpha)) \rangle \epsilon_B(1 \times U(\alpha)) \rangle,$$

and the commutative diagram

$$\begin{array}{ccc}
 U(B)^{\mathbf{m}} \times U(B) & \xleftarrow{\psi_{0(A,B,\alpha)}} & U(B)^{\mathbf{m}} \times U(B) \\
 \downarrow U(\beta)^{\mathbf{m}} \times U(\beta) & & \downarrow U(\beta)^{\mathbf{m}} \times U(\beta) \\
 U(A)^{\mathbf{m}} \times U(A) & \xleftarrow{\psi_{0\Delta(A)}} & U(A)^{\mathbf{m}} \times U(A) \\
 \downarrow U(\alpha)^{\mathbf{m}} \times U(\alpha) & & \downarrow U(\alpha)^{\mathbf{m}} \times U(\alpha) \\
 U(B)^{\mathbf{m}} \times U(B) & \xleftarrow{\psi_{0\Delta(B)}} & U(B)^{\mathbf{m}} \times U(B) \\
 \parallel & \nearrow \langle \eta_B, \epsilon_B \rangle & \\
 U(B)^{\mathbf{m}} \times U(B) & & 
 \end{array}$$

show the relationships between  $\gamma$  and  $\bar{\gamma}$ , and  $\gamma$  and  $\sigma$ ,  $\eta$  and  $\epsilon$ .  $\square$

From the two lemmas above we easily prove the following corollaries.

**Corollary 1.3.** *Each of the following types of data uniquely determine each other:*

- (a) a natural transformation  $\tau : V \rightarrow W$  with  $1_{D_{\mathbb{X}}} \circ \tau = 1_{D_{\mathbb{A}}^{\mathbf{m}} \times D_{\mathbb{A}}}$ ;
- (b) a natural transformation  $\rho : (U^{\mathbf{n}} \times U^{\mathbf{m}}) \times U \rightarrow U$  with component at each  $X$  in  $\mathbb{C}$  making the diagram

$$\begin{array}{ccc}
 (U(X)^{\mathbf{n}} \times U(X)^{\mathbf{m}}) \times U(X) & \xrightarrow{\rho_X} & U(X) \\
 \langle \theta_X, 1 \rangle \times 1 \uparrow & \nearrow \pi_2 & \\
 U(X)^{\mathbf{m}} \times U(X) & & 
 \end{array} \tag{11}$$

commute.

**Corollary 1.4.** *Each of the following types of data uniquely determine each other:*

- (a) a natural transformation  $\gamma : W \rightarrow V$  with  $1_{D_{\mathbb{X}}} \circ \gamma = 1_{D_{\mathbb{A}}^{\mathbf{m}} \times D_{\mathbb{A}}}$ ;
- (b) a natural transformation  $\sigma : U^{\mathbf{m}} \times (U \times U) \rightarrow U^{\mathbf{n}}$  with component at each  $X$  in  $\mathbb{C}$  making the diagram

$$\begin{array}{ccc}
 U(X)^{\mathbf{m}} \times (U(X) \times U(X)) & \xrightarrow{\sigma_X} & U(X)^{\mathbf{n}} \\
 1 \times \langle 1, 1 \rangle \uparrow & & \uparrow \theta_X \\
 U(X)^{\mathbf{m}} \times U(X) & \xrightarrow{\pi_1} & U(X)^{\mathbf{m}}
 \end{array} \tag{12}$$

commute.



**Corollary 1.5.** *Each of the following types of data uniquely determine each other:*

- (a) *natural transformations  $\tau : V \rightarrow W$  and  $\gamma : W \rightarrow V$  with  $1_{D_{\mathbb{X}}} \circ \tau = 1_{D_{\mathbb{A}}^{\mathbf{m}} \times D_{\mathbb{A}}}$  and  $1_{D_{\mathbb{X}}} \circ \gamma = 1_{D_{\mathbb{A}}^{\mathbf{m}} \times D_{\mathbb{A}}}$  and such that  $\tau\gamma = 1_W$ ;*
- (b) *natural transformations  $\rho : (U^{\mathbf{n}} \times U^{\mathbf{m}}) \times U \rightarrow U$  and  $\sigma : U^{\mathbf{m}} \times (U \times U) \rightarrow U^{\mathbf{n}}$  with components at each  $X$  in  $\mathbb{C}$  making the diagrams (11), (12) and*

$$\begin{array}{ccc}
 U(X)^{\mathbf{m}} \times (U(X) \times U(X)) & & \\
 \langle \langle \sigma, \pi_1 \rangle, \pi_2 \pi_2 \rangle \downarrow & \searrow \pi_1 \pi_2 & \\
 (U(X)^{\mathbf{n}} \times U(X)^{\mathbf{m}}) \times U(X) & \xrightarrow{\rho_X} & U(X)
 \end{array} \tag{13}$$

*commute.*

**Corollary 1.6.** *Each of the following types of data uniquely determine each other:*

- (a) *natural transformations  $\tau : V \rightarrow W$  and  $\gamma : W \rightarrow V$  with  $1_{D_{\mathbb{X}}} \circ \tau = 1_{D_{\mathbb{A}}^{\mathbf{m}} \times D_{\mathbb{A}}}$  and  $1_{D_{\mathbb{X}}} \circ \gamma = 1_{D_{\mathbb{A}}^{\mathbf{m}} \times D_{\mathbb{A}}}$  and such that  $\gamma\tau = 1_V$ ;*
- (b) *natural transformations  $\rho : (U^{\mathbf{n}} \times U^{\mathbf{m}}) \times U \rightarrow U$  and  $\sigma : U^{\mathbf{m}} \times (U \times U) \rightarrow U^{\mathbf{n}}$  with components at each  $X$  in  $\mathbb{C}$  making the diagrams (11), (12) and*

$$\begin{array}{ccc}
 (U(X)^{\mathbf{n}} \times U(X)^{\mathbf{m}}) \times U(X) & & \\
 \langle \pi_2 \pi_1, \langle \rho_X, \pi_2 \rangle \rangle \downarrow & \searrow \pi_1 \pi_1 & \\
 U(X)^{\mathbf{m}} \times (U(X) \times U(X)) & \longrightarrow & U(X)^{\mathbf{n}}
 \end{array} \tag{14}$$

*commute.*

**Corollary 1.7.** *Each of the following types of data uniquely determine each other:*

- (a) *natural transformations  $\tau : V \rightarrow W$  and  $\sigma : W \rightarrow V$  with  $1_{D_{\mathbb{X}}} \circ \tau = 1_{D_{\mathbb{A}}^{\mathbf{m}} \times D_{\mathbb{A}}}$  and  $1_{D_{\mathbb{X}}} \circ \sigma = 1_{D_{\mathbb{A}}^{\mathbf{m}} \times D_{\mathbb{A}}}$  and inverse to each other;*
- (b) *natural transformations  $\rho : (U^{\mathbf{n}} \times U^{\mathbf{m}}) \times U \rightarrow U$  and  $\sigma : U^{\mathbf{m}} \times (U \times U) \rightarrow U^{\mathbf{n}}$  with components at each  $X$  in  $\mathbb{C}$  making the diagrams (11), (12), (13) and (14) commute.*

We now consider the case where  $\mathbf{m} = \emptyset$  and  $\mathbf{n} = \{1\}$ , the results proved here will be used in Section 2.

When  $\mathbf{m} = \emptyset$  and  $\mathbf{n} = \{1\}$ , the functors  $V$  and  $W$  are up to isomorphism the functors  $\tilde{V}, \tilde{W} : \mathbf{Pt}(\mathbb{A}) \rightarrow \mathbf{Pt}(\mathbb{X})$  sending  $(A, B, \alpha, \beta)$  to

$$((U(A)_{\langle U(\alpha), \theta_B \rangle}^\times 1) \times U(B), U(B), \pi_2, \langle \langle \theta_A, 1 \rangle!_{U(B)}, 1 \rangle)$$

and

$$(U(A), U(B), U(\alpha), U(\beta))$$

respectively.

**Corollary 1.8.** *Each of the following types of data uniquely determine each other:*

- (a) a natural transformation  $\tau : \tilde{V} \rightarrow \tilde{W}$  with  $1_{D_{\mathbb{X}}} \circ \tau = 1_{D_{\mathbb{A}}}$  and with component at each  $(A, B, \alpha, \beta)$  in  $\mathbf{Pt}(\mathbb{A})$  such that the diagram

$$\begin{array}{ccc} U(A)_{\langle U(\alpha), \theta_B \rangle}^\times 1 & \xrightarrow{\langle 1, \theta_B! \rangle} & (U(A)_{\langle U(\alpha), \theta_B \rangle}^\times 1) \times U(B) \\ \parallel & & \downarrow \varphi_{1(A, B, \alpha, \beta)} \\ U(A)_{\langle U(\alpha), \theta_B \rangle}^\times 1 & \xrightarrow{\pi_1} & U(A) \end{array} \quad (15)$$

commutes;

- (b) a natural transformation  $\rho : U \times U \rightarrow U$  with component at each  $X$  in  $\mathbb{A}$  making the diagram

$$\begin{array}{ccc} & U(X) & \\ & \downarrow \langle 1, \theta_X! \rangle & \searrow 1_{U(X)} \\ U(X) \times U(X) & \xrightarrow{\rho_X} & U(X) \\ & \uparrow \langle \theta_X!, 1 \rangle & \nearrow 1_{U(X)} \\ & U(X) & \end{array} \quad (16)$$

commute.

**Corollary 1.9.** *Each of the following types of data uniquely determine each other:*

- (a) a natural transformation  $\gamma : \tilde{W} \rightarrow \tilde{V}$  with  $1_{D_{\mathbb{X}}} \circ \gamma = 1_{D_{\mathbb{A}}}$  and with component at each  $(A, B, \alpha, \beta)$  in  $\mathbf{Pt}(\mathbb{A})$  such that the diagram

$$\begin{array}{ccc} U(A)_{\langle U(\alpha), \theta_B \rangle}^\times 1 & \xrightarrow{\langle 1, \theta_B! \rangle} & (U(A)_{\langle U(\alpha), \theta_B \rangle}^\times 1) \times U(B) \\ \parallel & & \uparrow \psi_{1(A, B, \alpha, \beta)} \\ U(A)_{\langle U(\alpha), \theta_B \rangle}^\times 1 & \xrightarrow{\pi_1} & U(A) \end{array} \quad (17)$$

commutes;

- (b) a natural transformation  $\sigma : U \times U \rightarrow U$  with component at each  $X$  in  $\mathbb{A}$  making the diagram

$$\begin{array}{ccc}
 U(X) & & \\
 \langle 1, \theta_X ! \rangle \downarrow & \searrow 1_{U(X)} & \\
 U(X) \times U(X) & \xrightarrow{\sigma_X} & U(X) \\
 \langle 1, 1 \rangle \uparrow & & \uparrow \theta_X \\
 U(X) & \xrightarrow{!_{U(X)}} & 1
 \end{array} \tag{18}$$

commute.

**Corollary 1.10.** *Each of the following types of data uniquely determine each other:*

- (a) natural transformations  $\tau : \tilde{V} \rightarrow \tilde{W}$  and  $\gamma : \tilde{W} \rightarrow \tilde{V}$  with  $1_{D_{\mathbb{X}}} \circ \tau = 1_{D_{\mathbb{A}}}$  and  $1_{D_{\mathbb{X}}} \circ \gamma = 1_{D_{\mathbb{A}}}$  inverse to each other and with components at each  $(A, B, \alpha, \beta)$  in  $\mathbf{Pt}(\mathbb{A})$  making the diagrams (15) and (17) commute;
- (b) natural transformations  $\rho : U \times U \rightarrow U$  and  $\sigma : U \times U \rightarrow U$  with component at each  $X$  in  $\mathbb{A}$  making the diagrams (16), (18),

$$\begin{array}{ccc}
 U(X) \times U(X) & & \\
 \langle \sigma_X, \pi_2 \rangle \downarrow & \searrow \pi_1 & \\
 U(X) \times U(X) & \xrightarrow{\rho_X} & U(X)
 \end{array} \tag{19}$$

and

$$\begin{array}{ccc}
 U(X) \times U(X) & & \\
 \langle \rho_X, \pi_2 \rangle \downarrow & \searrow \pi_1 & \\
 U(X) \times U(X) & \xrightarrow{\sigma_X} & U(X)
 \end{array} \tag{20}$$

commute.

In the sections that follows we use the fact that the set of natural transformation  $U^{\mathbf{n}} \rightarrow U$  (where  $\mathbf{n} = \{1, \dots, n\}$  and  $U$  is the forgetful functor from a variety to sets) is in bijection with the set of  $n$ -ary terms of the variety. Since this is no longer true for arbitrary internal varieties (every term determines a natural transformation but not conversely) the results in the sections that follow hold only partially in arbitrary internal varieties, i.e. the existence of certain terms determine natural transformations between appropriate  $V$  and  $W$  but not conversely.

## 2 Pointed varieties

In this section we apply the results from Section 1 to the special case where  $\mathbb{A} = \mathcal{V}$  is a pointed variety,  $\mathbb{X} = \mathbf{Set}_*$  is the category of pointed sets,  $U$  is the usual forgetful functor,  $\mathbf{m} = \emptyset$ ,  $\mathbf{n} = \{1\}$ , and  $\theta$  is constructed using the constant of  $\mathcal{V}$ .

For any category  $\mathbb{C}$  we define  $\mathbf{SplExt}(\mathbb{C})$  to be the category of split extensions: an object is a sextuple  $(K, A, B, \kappa, \alpha, \beta)$  where  $K, A$  and  $B$  are objects in  $\mathbb{C}$  and  $\kappa : K \rightarrow B$ ,  $\alpha : A \rightarrow B$  and  $\beta : B \rightarrow A$  are morphisms in  $\mathbb{C}$  with  $(K, \kappa)$  the kernel of  $\alpha$  and  $\alpha\beta = 1_B$ ; a morphism  $(K, A, B, \kappa, \alpha, \beta) \rightarrow (K', A', B', \kappa', \alpha', \beta')$  is a triple  $(u, v, w)$  of morphisms  $u : K \rightarrow K'$ ,  $v : A \rightarrow A'$  and  $w : B \rightarrow B'$  such that in the diagram

$$\begin{array}{ccccc} K & \xrightarrow{\kappa} & A & \xrightleftharpoons[\beta]{\alpha} & B \\ u \downarrow & & v \downarrow & & w \downarrow \\ K' & \xrightarrow{\kappa'} & A' & \xrightleftharpoons[\beta']{\alpha'} & B' \end{array}$$

$$v\kappa = \kappa'u, \alpha'v = w\alpha \text{ and } v\beta = \beta'w.$$

**Theorem 2.1.** *Let  $\mathcal{V}$  be a pointed variety and let  $P, Q : \mathbf{SplExt}(\mathcal{V}) \rightarrow \mathbf{SplExt}(\mathbf{Set}_*)$  be the functors taking  $(K, A, B, \kappa, \alpha, \beta)$  to  $(U(K), U(K) \times U(B), \langle 1, 0 \rangle, \pi_2, \langle 0, 1 \rangle)$  and  $(U(K), U(A), U(B), U(\kappa), U(\alpha), U(\beta))$  respectively.*

(a)  *$\mathcal{V}$  is a unital variety [1] if and only if there exists a natural transformation  $P \rightarrow Q$  with component at  $(K, A, B, \kappa, \alpha, \beta)$  of the form*

$$\begin{array}{ccccc} U(K) & \longrightarrow & U(K) \times U(B) & \xrightleftharpoons[\langle 0, 1 \rangle]{\pi_2} & U(B) \\ \parallel & & \downarrow & & \parallel \\ U(K) & \longrightarrow & U(A) & \xrightleftharpoons[U(\beta)]{U(\alpha)} & U(B); \end{array}$$

(b)  *$\mathcal{V}$  is a subtractive variety [6] if and only if there exists a natural transformation  $Q \rightarrow P$  with component at  $(K, A, B, \kappa, \alpha, \beta)$  of the form*

$$\begin{array}{ccccc} U(K) & \longrightarrow & U(A) & \xrightleftharpoons[U(\beta)]{U(\alpha)} & U(B) \\ \parallel & & \downarrow & & \parallel \\ U(K) & \longrightarrow & U(K) \times U(B) & \xrightleftharpoons[\langle 0, 1 \rangle]{\pi_2} & U(B); \end{array}$$

(c)  $\mathcal{V}$  is a variety of right  $\Omega$ -loops if and only if there exists a natural isomorphism  $P \rightarrow Q$  with component at  $(K, A, B, \kappa, \alpha, \beta)$  of the form

$$\begin{array}{ccccc} U(K) & \longrightarrow & U(K) \times U(B) & \xrightleftharpoons[\langle 0,1 \rangle]{\pi_2} & U(B) \\ \parallel & & \downarrow & & \parallel \\ U(K) & \longrightarrow & U(A) & \xrightleftharpoons[U(\beta)]{U(\alpha)} & U(B). \end{array}$$

*Proof.* It is easy to see that to give a natural transformation  $P \rightarrow Q$  as in (a) above is the same as to give a natural transformation  $\tilde{V} \rightarrow \tilde{W}$  as in (a) of Corollary 1.8 which, by Corollary 1.8, is uniquely determined by a natural transformation  $\rho : U \times U \rightarrow U$  with components making the diagram (16) commute. And, such a natural transformation determines and is determined by a binary term  $+$  such that for each  $x, y$  in  $X$ , an algebra,  $x + y = \rho_X(x, y)$ . The commutativity of (16) then implies that  $x + 0 = x = 0 + x$ . The statements (b) and (c) follow from Corollaries 1.9, and 1.10 in a similar way.  $\square$

**Corollary 2.2.** *Let  $\tilde{P}, \tilde{Q} : \mathbf{Pt}(\mathbb{A}) \rightarrow \mathbf{Pt}(\mathbb{X})$  be the functors sending  $(A, B, \alpha, \beta)$  in  $\mathbf{Pt}(\mathbb{A})$  to  $(U(K \times B), U(B), U(\pi_2), U(\langle 0, 1 \rangle))$  (where  $K = \text{Ker}(\alpha)$ ) and  $(U(A), U(B), U(\alpha), U(\beta))$  respectively.  $\mathcal{V}$  is a variety of right  $\Omega$ -loops if and only if there exists a natural bijection  $\tilde{P} \rightarrow \tilde{Q}$  with component  $(A, B, \alpha, \beta)$  of the form*

$$\begin{array}{ccc} U(K \times B) & \xrightleftharpoons[U(\langle 0,1 \rangle)]{U(\pi_2)} & U(B) \\ \downarrow & & \parallel \\ U(A) & \xrightleftharpoons[U(\beta)]{U(\alpha)} & U(B). \end{array}$$

*Proof.* It follows from Corollary 1.7 that a natural bijection  $\tilde{P} \rightarrow \tilde{Q}$  as above is completely determined by and determines binary terms  $\rho(x, y)$  and  $\sigma(x, y)$  satisfying the identities  $\sigma(x, x) = 0$ ,  $\rho(\sigma(x, y), y) = x$  and  $\sigma(\rho(x, y), y) = x$ . Setting  $x + y = \rho(\sigma(x, 0), y)$  and  $x - y = \rho(\sigma(x, y), 0)$  determines terms that satisfy the right loop identities.  $\square$

**Remark 2.3.** *In fact it can be shown that  $\mathcal{V}$  is a variety of right  $\Omega$ -loops if and only if there exists a natural isomorphism  $\tilde{P} \rightarrow \tilde{Q}$ .*

### 3 Protomodular varieties

In this section we give a new classification of protomodular varieties by applying the results from Section 1 to the case where  $\mathbb{A} = \mathcal{V}$  is an arbitrary variety with constants,  $\mathbb{X} = \mathbf{Set}$  is the category of sets, and  $U$  is the usual forgetful functor.

**Theorem 3.1.**  $\mathcal{V}$  is a protomodular variety if and only if for some  $\mathbf{m} = \{1, \dots, m\}$ ,  $\mathbf{n} = \{1, \dots, n\}$  and  $\theta$  there exist natural transformations  $\tau : V \rightarrow W$  and  $\gamma : W \rightarrow V$  with  $\tau\gamma = 1_W$  and with components at each  $(A, B, \alpha, \beta)$  in  $\mathbf{Pt}(\mathbf{C})$  of the form

$$\begin{array}{ccc} (U(A)^{\mathbf{n}}_{\langle U(\alpha)^{\mathbf{n}}, \theta_B \rangle} U(B)^{\mathbf{m}}) \times U(B) & \xrightleftharpoons[\langle U(\beta)^{\mathbf{m}}, \theta_B, 1 \rangle \times 1]{\pi_2 \times 1} & U(B)^{\mathbf{m}} \times U(B) \\ \uparrow \downarrow & & \parallel \\ U(B)^{\mathbf{m}} \times U(A) & \xrightleftharpoons[1 \times U(\beta)]{1 \times U(\alpha)} & U(B)^{\mathbf{m}} \times U(B). \end{array}$$

*Proof.* It follows from Corollary 1.5 that natural transformations  $\tau : V \rightarrow W$  and  $\gamma : W \rightarrow V$  as above determine terms

$$\rho(x_1, \dots, x_n, y_1, \dots, y_m, z) \text{ and } \sigma_i(y_1, \dots, y_m, x, z) \ i \in \mathbf{n}$$

satisfying the identities

$$\sigma_i(y_1, \dots, y_m, x, x) = \theta_i(y_1, \dots, y_m) \ i \in \mathbf{n}$$

$$\rho(\sigma_1(y_1, \dots, y_m, x, z), \dots, \sigma_n(y_1, \dots, y_m, x, z), y_1, \dots, y_m, z) = x.$$

For any constant  $e$  we may form new terms  $e_i = \theta_i(e, \dots, e) \ i \in \mathbf{n}$ ,  $s_i(x, z) = \sigma_i(e, \dots, e, x, z) \ i \in \mathbf{n}$ , and  $p(x_1, \dots, x_n, z) = \rho(x_1, \dots, x_n, e, \dots, e, z)$ . It is easy to check that these terms make  $\mathcal{V}$  a protomodular variety. The converse follows from Corollary 1.5 with  $\mathbf{m} = \emptyset$ .  $\square$

**Remark 3.2.** The results in this section can easily be extended to  $\mathcal{V}$  an infinitary variety, with  $\mathbf{m}$  and  $\mathbf{n}$  possibly infinite sets, giving, by Theorem 2.1 of [3], a new classification of infinitary protomodular varieties.

**Remark 3.3.** It could also be interesting to study when  $\gamma\tau = 1_V$  (without  $\tau\gamma = 1_W$ ) which can be seen to be equivalent to the existence of  $\rho$  and  $\sigma$  as above, satisfying the identities:

$$\begin{aligned} \sigma_i(y_1, \dots, y_m, x, x) &= \theta_i(y_1, \dots, y_m) \ i \in \mathbf{n} \\ \rho(\theta_1(y_1, \dots, y_m), \dots, \theta_n(y_1, \dots, y_m), y_1, \dots, y_m, x) &= x \\ \sigma_i(y_1, \dots, y_m, \rho(x_1, \dots, x_n, y_1, \dots, y_m, z), z) &= x_i \ i \in \mathbf{n} \end{aligned}$$

instead.

## 4 General varieties

In this section we consider the case where  $\mathbb{A} = \mathcal{V}$  is a variety,  $\mathbb{X} = \mathbf{Set}$  is the category sets, and  $U$  is the usual forgetful functor.

For a variety  $\mathcal{V}$  consider the condition:

**Condition 4.1.** *There exist ternary terms  $p$  and  $q$  satisfying the identities:  $p(x, x, y) = y$  and  $p(q(x, y, z), z, y) = x = q(p(x, y, z), z, y)$ .*

It is easy to see that  $q(x, x, y) = y$  follows from the conditions above, as remarked in [7], where such a variety was called a *biternary system*.

**Remark 4.2.** *It is easy to see that if a variety  $\mathcal{V}$  satisfies Condition 4.1 then every regular epimorphism  $f : E \rightarrow B$  is up to bijection a product projection  $\pi_2 : X \times B \rightarrow B$  for some  $X$  (since for each  $b$  and  $b'$  choosing  $e$  and  $e'$  in  $f^{-1}(\{b\})$  and  $f^{-1}(\{b'\})$  respectively gives a bijection  $p(-, e, e') : f^{-1}(\{b\}) \rightarrow f^{-1}(\{b'\})$ ).*

**Proposition 4.3.** *For a variety  $\mathcal{V}$  the following conditions are equivalent:*

1.  $\mathcal{V}$  satisfies Condition 4.1;
2. *There exist ternary terms  $\tilde{p}$  and  $\tilde{q}$  satisfying the identities:  $\tilde{p}(x, x, y) = y = \tilde{q}(x, x, y)$ ,  $\tilde{p}(x, y, y) = x = \tilde{q}(x, y, y)$  and  $\tilde{p}(\tilde{q}(x, y, z), z, y) = x = \tilde{q}(\tilde{p}(x, y, z), z, y)$ ;*
3. *There exists a quaternary term  $u$  satisfying the identities:  $u(a, b, b, a) = b$  and  $u(u(a, b, c, d), b, d, c) = a$ ;*
4. *There exists a quaternary term  $\tilde{u}$  satisfying the identities:  $\tilde{u}(a, b, b, a) = b = \tilde{u}(a, a, b, a)$  and  $\tilde{u}(a, b, c, c) = a = \tilde{u}(\tilde{u}(a, b, c, d), b, d, c)$ ;*

*If in addition  $\mathcal{V}$  has at least one constant, those conditions are further equivalent to:*

5. *For each constant  $e$  there exist binary terms  $x + y$  and  $x - y$  satisfying the right loop identities (for that constant  $e$ ).*

*Proof.* The implications  $2 \Rightarrow 1$  and  $4 \Rightarrow 3$  are trivial.

$1 \Rightarrow 2$  : Given  $p$  and  $q$  define  $\tilde{p}(x, y, z) = p(q(x, y, y), y, z)$  and  $\tilde{q}(x, y, z) = p(q(x, y, z), z, z)$ .

$2 \Rightarrow 4$  : Given  $\tilde{p}$  and  $\tilde{q}$  define  $\tilde{u}(a, b, c, d) = \tilde{p}(\tilde{q}(a, b, c), d, b)$ .

$3 \Rightarrow 1$  : Given  $u$  define  $p(x, y, z) = u(x, z, z, y)$  and  $q(x, y, z) = u(x, y, z, y)$ .

If in addition  $\mathcal{V}$  has at least one constant.

$2 \Rightarrow 5$  : Given  $\tilde{p}$  and  $\tilde{q}$  for each constant  $e$  define  $x + y = \tilde{p}(x, e, y)$  and  $x - y = \tilde{q}(x, y, e)$ .

$5 \Rightarrow 1$  : Given  $x + y$  and  $x - y$  for some constant  $e$  define  $p(x, y, z) = q(x, y, z) = (x - y) + z$ .  $\square$

**Remark 4.4.** *It follows that a variety satisfying Condition 4.1 is a Mal'tsev variety.*

**Theorem 4.5.** (a) If  $\mathcal{V}$  satisfies Condition 4.1, then for  $\mathbf{n} = \{1\}$ ,  $\mathbf{m} = \{1\}$  and  $\theta = 1_U$  there exists a natural isomorphism  $\tau : V \rightarrow W$  with component at each  $(A, B, \alpha, \beta)$  in  $\mathbf{Pt}(\mathbb{C})$  of the form

$$\begin{array}{ccc} (U(A)^{\mathbf{n}}_{\langle U(\alpha)^{\mathbf{n}}, \theta_B \rangle}) U(B)^{\mathbf{m}} \times U(B) & \xrightleftharpoons[\langle U(\beta)^{\mathbf{m}} \theta_{B,1} \rangle \times 1]{\pi_2 \times 1} & U(B)^{\mathbf{m}} \times U(B) \\ \downarrow & & \parallel \\ U(B)^{\mathbf{m}} \times U(A) & \xrightleftharpoons[1 \times U(\beta)]{1 \times U(\alpha)} & U(B)^{\mathbf{m}} \times U(B); \end{array}$$

(b) If for some  $\mathbf{n} = \{1, \dots, n\}$ ,  $\mathbf{m} = \{1, \dots, m\}$  and  $\theta$  there exists a natural isomorphism  $\tau : V \rightarrow W$  with component at each  $(A, B, \alpha, \beta)$  in  $\mathbf{Pt}(\mathbb{C})$  of the form

$$\begin{array}{ccc} (U(A)^{\mathbf{n}}_{\langle U(\alpha)^{\mathbf{n}}, \theta_B \rangle}) U(B)^{\mathbf{m}} \times U(B) & \xrightleftharpoons[\langle U(\beta)^{\mathbf{m}} \theta_{B,1} \rangle \times 1]{\pi_2 \times 1} & U(B)^{\mathbf{m}} \times U(B) \\ \downarrow & & \parallel \\ U(B)^{\mathbf{m}} \times U(A) & \xrightleftharpoons[1 \times U(\beta)]{1 \times U(\alpha)} & U(B)^{\mathbf{m}} \times U(B), \end{array}$$

then  $\mathcal{V}$  satisfies Condition 4.1.

*Proof.* (a) Let  $\mathbf{n} = \mathbf{m} = \{1\}$  and  $\theta = 1_U$ . Given ternary terms  $p$  and  $q$  as in Condition 4.1, it is easy to check that  $\rho = p$  and  $\sigma(x, y, z) = q(y, z, x)$  define natural transformations making the diagrams (11), (12), (13) and (14) commute. Therefore by Corollary 1.7 determine a natural isomorphism  $V \rightarrow W$ , as required.

(b) If for some  $\mathbf{n} = \{1, \dots, n\}$ ,  $\mathbf{m} = \{1, \dots, m\}$  and  $\theta$  there exists an isomorphism  $V \rightarrow W$  then by Corollary 1.7 there exist terms  $\rho(x_1, \dots, x_n, y_1, \dots, y_m, z)$  and  $\sigma_i(y_1, \dots, y_m, x, z)$   $i \in \mathbf{n}$  satisfying the identities:

$$\begin{aligned} \sigma_i(y_1, \dots, y_m, x, x) &= \theta_i(y_1, \dots, y_m) \\ \rho(\sigma_1(y_1, \dots, y_m, x, z), \dots, \sigma_n(y_1, \dots, y_m, x, z), y_1, \dots, y_m, z) &= x \\ \sigma_i(y_1, \dots, y_m, \rho(x_1, \dots, x_n, y_1, \dots, y_m, z), z) &= x_i. \end{aligned}$$

Let  $p$  and  $q$  be the terms defined by

$$\begin{aligned} p(x, y, z) &= \rho(\sigma_1(y, \dots, y, x, y), \dots, \sigma_n(y, \dots, y, x, y), y, \dots, y, z) \\ q(x, y, z) &= \rho(\sigma_1(z, \dots, z, x, y), \dots, \sigma_n(z, \dots, z, x, y), z, \dots, z, z). \end{aligned}$$

It is easy to check that  $p$  and  $q$  satisfy the desired identities as in Condition 4.1. □



Recall that for any category  $\mathbb{C}$  the functor  $D_{\mathbb{C}}$  is the functor  $\mathbf{Pt}(\mathbb{C}) \rightarrow \mathbb{C}$  taking  $(A, B, \alpha, \beta)$  to  $B$ .

**Theorem 4.6.** *Let  $V$  and  $W$  be the functors defined in Section 1 with  $\mathbb{A} = \mathcal{V}$ ,  $\mathbb{X} = \mathbf{Set}$ ,  $U$  the usual forgetful functor,  $\mathbf{n} = \mathbf{m} = \{1\}$  and  $\theta = 1_U$ . Let  $P, Q : (\mathbb{A} \downarrow D_{\mathbb{A}}) \rightarrow \mathbf{Pt}(\mathbb{X})$  be the functors sending  $(E, (A, B, \alpha, \beta), f)$  to*

$$(U(A \times_{\langle \alpha, f \rangle} E) \times U(B), U(E) \times U(B), U(\pi_2) \times 1, U(\langle \beta f, 1 \rangle) \times 1)$$

and

$$(U(E) \times U(A), U(E) \times U(B), 1 \times U(\alpha), 1 \times U(\beta))$$

respectively. The following are equivalent:

1. There exists an isomorphism  $\tau : V \rightarrow W$  with component at each  $(A, B, \alpha, \beta)$  in  $\mathbf{Pt}(\mathbb{A})$  of the form

$$\begin{array}{ccc} (U(A)^{\mathbf{n}} \times_{\langle U(\alpha)^{\mathbf{n}}, \theta_B \rangle} U(B)^{\mathbf{m}}) \times U(B) & \xrightleftharpoons[\langle U(\beta)^{\mathbf{m}} \theta_B, 1 \rangle \times 1]{\pi_2 \times 1} & U(B)^{\mathbf{m}} \times U(B) \\ \downarrow \Upsilon & & \parallel \\ U(B)^{\mathbf{m}} \times U(A) & \xrightleftharpoons[1 \times U(\beta)]{1 \times U(\alpha)} & U(B)^{\mathbf{m}} \times U(B); \end{array}$$

2. There exists an isomorphism  $\chi : P \rightarrow Q$  with component at each  $(E, (A, B, \alpha, \beta), f)$  in  $(\mathbb{A} \downarrow D_{\mathbb{A}})$  of the form

$$\begin{array}{ccc} (U(A \times_{\langle \alpha, f \rangle} E) \times U(B)) & \xrightleftharpoons[U(\langle \beta f, 1 \rangle) \times 1]{U(\pi_2) \times 1} & U(E) \times U(B) \\ \downarrow \Upsilon & & \parallel \\ U(E) \times U(A) & \xrightleftharpoons[1 \times U(\beta)]{1 \times U(\alpha)} & U(E) \times U(B); \end{array}$$

3.  $\mathcal{V}$  satisfies Condition 4.1.

*Proof.* The equivalence of 1 and 3 follows from Theorem 4.5. It is easy to show that  $2 \Rightarrow 1$  since  $P$  and  $Q$  composed with the functor sending  $(A, B, \alpha, \beta)$  in  $\mathbf{Pt}(\mathbb{A})$  to  $(B, (A, B, \alpha, \beta), 1_B)$  in  $(\mathbb{A} \downarrow D_{\mathbb{A}})$  are up to natural isomorphism the functors  $V$  and  $W$  respectively. We will show that  $3 \Rightarrow 2$ .

Let  $p$  and  $q$  be ternary terms as in Condition 4.1. It is easy to check that  $\chi$  with component at each  $(E, (A, B, \alpha, \beta), f)$  defined by  $\chi_{(E, (A, B, \alpha, \beta), f)} = (\varphi_{(E, (A, B, \alpha, \beta), f)}, 1_{U(B)})$  where  $\varphi_{(E, (A, B, \alpha, \beta), f)}((a, e), b) = (e, p(a, \beta f(e), \beta(b)))$  is an isomorphism with inverse  $\chi_{(E, (A, B, \alpha, \beta), f)}^{-1} = (\psi_{(E, (A, B, \alpha, \beta), f)}, 1_{U(B)})$  where  $\psi_{(E, (A, B, \alpha, \beta), f)}(e, a) = ((q(a, \beta \alpha(a), \beta f(e)), e), \alpha(a))$ .  $\square$

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